

19/

ON NON-LINEAR BOUNDARY VALUE TYPE PROBLEMS

by

R. Conti \*\*

164-24019  
CODE - 1  
NASA CR 56652  
Cat. 20

OTS PRICE

Introduction

The title refers to problems

XEROX

\$

1.60 ph

MICROFILM

\$

X

(S)

$$L_1 x = H_1 x,$$

$$L_2 x = H_2 x$$

where the linear operators  $L_1, L_2$  and the non-necessarily linear operators  $H_1, H_2$  are defined in a linear space  $X$  and have values in two linear spaces  $X_1, X_2$  respectively.

In Sections 1 and 2 we first discuss some assumptions which allow us to reduce (S) to a single equation

(T)

$$x = Tx \quad .$$

In Sections 3 and 4 some existence criteria for (S), via (T), are given.

As an application, a boundary value problem for an ordinary differential equation is considered in Sections 5 and 6.

The presentation of these topics, originally inspired by a preceding paper of mine (R. Conti, [2]), has been substantially modified and considerably

\* This research was supported in part by the United States Air Force through the Air Force Office of Scientific Research, Office of Aerospace Research, under Contract No. AF 49(638)-1242, in part by the National Aeronautics and Space Administration under Contract No. NASw-845 and in part by the Office of Naval Research under Contract No. Nonr-3693(00).

\*\*Research Institute for Advanced Studies (RIAS) and University of Florence, Florence, Italy.

improved, as a result of discussions in a Seminar given at RIAS in February 1964. In particular I am indebted to Dr. Jack Hale for various helpful suggestions and criticisms.

1. There are various problems in Analysis which can be formulated in the following way:

Let us denote by  $X, X_1, X_2$  three linear vector spaces. Given two linear operators

$$L_1: X \rightarrow X_1, \quad L_2: X \rightarrow X_2$$

and two (non-necessarily linear) operators

$$H_1: X \rightarrow X_1, \quad H_2: X \rightarrow X_2$$

we want to solve the system of equations

$$(S) \quad L_1 x = H_1 x, \quad L_2 x = H_2 x$$

Many particular cases of this problem show that it is not unrealistic to assume that one of the two linear operators, say  $L_1$ , maps  $X$  onto the whole of  $X_1$ . This is equivalent to the hypothesis that

1)  $L_1$  has a right inverse, i.e. there exists some linear operator

$$\tilde{L}_1: X_1 \rightarrow X$$

such that

$$(1) \quad L_1 \tilde{L}_1 = I_1$$

where  $I_1$  is the identity operator in  $X_1$ .

Let us now examine the effect of this assumption. Let  $x$  be a solution of (S). Therefore  $L_1 x = H_1 x$ , and, by 1),  $L_1(x - \tilde{L}_1 H_1 x) = 0_1$ , where  $0_1$  is the zero element of  $X_1$ , so that  $y = x - \tilde{L}_1 H_1 x \in \mathcal{N}[L_1]$ , the null space of  $L_1$ . Denoting by  $L_{2,0}$  the restriction of  $L_2$  to  $\mathcal{N}[L_1]$  we shall have  $L_{2,0} y = L_{2,0}(x - \tilde{L}_1 H_1 x) = L_2(x - \tilde{L}_1 H_1 x) = L_2 x - L_2 \tilde{L}_1 H_1 x$ , and since  $L_2 x = H_2 x$ , we also have

$$L_{2,0} y = (H_2 - L_2 \tilde{L}_1 H_1) x$$

i.e.  $L_{2,0} y = K_2 x$ , with

$$K_2 = H_2 - L_2 \tilde{L}_1 H_1: \quad X \rightarrow X_2$$

Since  $y, K_2 x$  are fixed elements of  $\mathcal{N}[L_1]$  and  $X_2$  respectively we may take some linear operator

$$A: X_2 \rightarrow \mathcal{N}[L_1]$$

such that  $y = AK_2 x$ . Therefore  $L_{2,0} AK_2 x = K_2 x$  and we conclude that if  $x$

is a solution of (S) then, under the assumption i) it is also a solution of

$$(S_A) \quad \begin{cases} x = \Lambda(H_2 - L_2 \tilde{L}_1 H_1)x + \tilde{L}_1 H_1 x, \\ (L_{2,0} \Lambda - L_2)(H_2 - L_2 \tilde{L}_1 H_1)x = 0_2 \end{cases}$$

for some linear operator  $\Lambda: X_2 \rightarrow \mathcal{H}[L_1]$ , where  $L_2$  is the identity operator in  $X_2$  and  $0_2$  is the zero element of  $X_2$ . Conversely if  $x$  is a solution of  $(S_A)$  for some  $\Lambda$ , we have  $L_1 x = L_1(\Lambda K_2 x + \tilde{L}_1 H_1 x) = L_1 \Lambda K_2 x + L_1 \tilde{L}_1 H_1 x = H_1 x$ , and  $L_2 x = L_2(\Lambda K_2 x + \tilde{L}_1 H_1 x) = L_{2,0} \Lambda K_2 x + L_2 \tilde{L}_1 H_1 x = K_2 x + L_2 \tilde{L}_1 H_1 x = H_2 x$ .

Therefore denoting by  $\Sigma$  the set of solutions of (S), by  $\Sigma_A$  the set of solutions of  $(S_A)$  and by  $\bigcup_A \Sigma_A$  the union of the sets  $\Sigma_A$  for all the linear operators  $\Lambda: X_2 \rightarrow \mathcal{H}[L_1]$  we have

Theorem 1: Under the assumption i),  $\Sigma = \bigcup_A \Sigma_A$ .

Remark 1. If  $H_1, H_2$  are constant operators

$$H_1 x = h_1, \quad H_2 x = h_2, \quad x \in X$$

then  $K_2$  is also constant

$$K_2 x = h_2 - L_2 \tilde{L}_1 h_1, \quad x \in X$$

and we have that, under i), the system

$$L_1 x = h_1, \quad L_2 x = h_2,$$

has solutions if and only if

$$h_2 - L_2 \tilde{L}_1 h_1 \in \mathcal{N}[L_{2,0} \Lambda - I_2]$$

for some  $\Lambda$ , or, equivalently,  $h_2 - L_2 \tilde{L}_1 h_1$  is orthogonal to the range of the adjoint operator

$$\Lambda' L_{2,0}' - I_2' = (L_{2,0} \Lambda - I_2)'; \quad x_2' \rightarrow x_2'.$$

2. Having transformed (S) into  $(S_\Lambda)$  we try to reduce  $(S_\Lambda)$  to the first equation. Therefore we look for those  $\Lambda$  which yield the largest possible  $\mathcal{N}[L_{2,0} \Lambda - I_2]$ . Denoting by  $K_2(X)$  the image of  $X$  under  $K_2$  we see that, for instance, if there exists such a  $\Lambda$  that

$$\mathcal{N}[L_{2,0} \Lambda - I_2] \supset K_2(X)$$

then the second equation in  $(S_\Lambda)$  is an identity so that it will be sufficient to solve the first equation of  $(S_\Lambda)$ , i.e.

$$(T) \quad x = \Lambda H_2 x + (-\Lambda L_2 \tilde{L}_1 + \tilde{L}_1) H_1 x$$

in order to have solutions of (S).

The preceding assumption is equivalent to

ii) There exists a linear operator  $\Lambda: X_2 \rightarrow \mathcal{N}[L_1]$  such that

$$(L_{2,0}\Lambda - I_2) (H_2 - L_2 \tilde{L}_1 H_1) = 0$$

where  $0$  is the zero operator in  $X$ .

In particular this will be satisfied if there exists a  $\Lambda$  such that  $L_{2,0}\Lambda - I_2 = 0_2$ , the zero operator in  $X_2$ , i.e. if

ii') The restriction  $L_{2,0}$  of  $L_2$  to  $\mathcal{N}[L_1]$  has a right inverse,  
i.e. there exists a linear operator  $\Lambda: X_2 \rightarrow \mathcal{N}[L_1]$  such that

$$L_{2,0} \Lambda = I_2$$

Still more in particular we may assume that

ii'') The restriction  $L_{2,0}$  of  $L_2$  to  $\mathcal{N}[L_1]$  has an inverse, i.e.  
there exists a linear operator

$$L_{2,0}^{-1}: X_2 \rightarrow \mathcal{N}[L_1]$$

such that

$$(2) \quad L_{2,0} L_{2,0}^{-1} = I_2$$

and also

$$(3) \quad L_{2,0}^{-1} L_{2,0} = I_0$$

where  $I_0$  is the identity operator in  $\mathcal{N}[L_1]$ .

In this case (T) can be written

$$(T_0) \quad x = L_{2,0}^{-1} H_2 x + (-L_{2,0}^{-1} L_2 \tilde{L}_1 + \tilde{L}_1) H_1 x$$

and then not only a solution of  $(T_0)$  is a solution of (S), but the converse is also true. In fact if  $x$  is a solution of (S) then  $y = x - \tilde{L}_1 H_1 x \in \mathcal{N}[L_1]$ , hence by (3),  $x - \tilde{L}_1 H_1 x = L_{2,0}^{-1} L_{2,0} (x - \tilde{L}_1 H_1 x) = L_{2,0}^{-1} L_2 (x - \tilde{L}_1 H_1 x) = L_{2,0}^{-1} L_2 x - L_{2,0}^{-1} \tilde{L}_1 H_1 x = L_{2,0}^{-1} H_2 x - L_{2,0}^{-1} \tilde{L}_1 H_1 x$ .

Summing up we have

Theorem 2: Under the hypothesis i), if ii) (or ii')) holds, every solution of (T) is also a solution of (S). If ii") holds then (S) is equivalent to  $(T_0)$ .

Remark 2. Since the homogeneous system

$$(S_0) \quad L_1 y = 0_1, \quad L_2 y = 0_2$$

can be written

$$L_{2,0} y = 0_2$$

(3) means that  $y = 0$  is the unique solution of  $(S_0)$ .

Remark 3. In general,  $ii''$ ) is more restrictive than  $ii'$ ). If, however,  $X_2$  and  $N[L_1]$  are both  $n$ -dimensional, then  $ii'$ ) and  $ii''$ ) are actually equivalent assumptions. In fact, denoting by  $\mathcal{T}$  any isomorphism of  $X_2$  onto  $N[L_1]$ ,  $L_{2,0}\mathcal{T}$  is an endomorphism of  $X_2$ . By  $ii'$ ) there exists  $\Lambda$  such that  $L_{2,0}\Lambda = I_2$ , hence  $L_{2,0}\mathcal{T}\mathcal{T}^{-1}\Lambda = I_2$ , so that  $\mathcal{T}^{-1}\Lambda$  is a right inverse of  $L_{2,0}\mathcal{T}$ . Since  $X_2$  is finite dimensional,  $\mathcal{T}^{-1}\Lambda$  will be a left inverse too, i.e.  $\mathcal{T}^{-1}\Lambda L_{2,0}\mathcal{T} = I_0$ , hence  $\Lambda L_{2,0} = I_0$ , and finally  $\Lambda = L_{2,0}^{-1}$ .

3. Theorem 2 is a reduction theorem which makes available for (S) the techniques known for solving the single equation (T). Such techniques, which include approximation methods, fixed point theorems, variational methods, etc. (See for instance, M. A. Krasnosel'skiĭ [1], [2]; H. Ehrmann [2]) require that the linear space  $X$  be made into some topological (or even normed) space and that some suitable continuity (or even complete continuity) assumptions be imposed on the operator

$$(4) \quad T = \Lambda H_2 + (-\Lambda L_2 \tilde{L}_1 + \tilde{L}_1) H_1 \quad .$$

In the next section we shall establish some existence criteria for (S) assuming, among others, that  $X$  is a subspace of a Banach space  $\overset{\Lambda}{X}$  and using classical fixed point theorems to solve (T).

Better results could certainly be obtained by using more sophisticated methods, especially if the Banach space  $\overset{\Delta}{X}$  is known to have some additional properties.

For instance if  $H_2$  is a constant operator, then (T) is a Hammerstein equation for which a fairly developed theory is now available if  $\overset{\Delta}{X}$  is a Hilbert space (see I.I Kolodner, [1], also H. Ehrmann [2]).

4. The first two existence criteria are based on Banach's contractive mapping principle.

Theorem 3. If 1) and ii) hold, and, in addition

iii)  $X$  is a subspace of a Banach space  $\overset{\Delta}{X}$ ;  $X_1, X_2$  are (linear) normed spaces;

iv)  $A, -\Lambda L_2 \tilde{L}_1 + \tilde{L}_1$  are bounded (linear) operators;

v)  $H_1, H_2$  can be extended to  $\overset{\Delta}{X}$  in such a way that

$$(5) \quad |H_1 x' - H_1 x''|_{X_1} \leq \lambda_1 |x' - x''|_{\overset{\Delta}{X}}, \quad x', x'' \in \overset{\Delta}{X}; \quad i = 1, 2,$$

with

$$(6) \quad |\Lambda| \lambda_2 + |-\Lambda L_2 \tilde{L}_1 + \tilde{L}_1| \lambda_1 < 1$$

then (S) has solutions.

In fact from (4), (5), (6) still denoting by  $T$  the extension of  $T$  to  $\hat{X}$ , it follows

$$\|Tx' - Tx''\|_{\hat{X}} \leq \alpha \|x' - x''\|_{\hat{X}}, \quad x', x'' \in \hat{X}$$

with  $\alpha < 1$ , hence, by Banach's theorem,  $(T)$  has one (and only one) solution in  $\hat{X}$ , but since by (4),  $T$  maps  $\hat{X}$  into  $X$ , this solution belongs to  $X$  and by Theorem 2,  $(S)$  has at least one solution.

Using the last statement of Theorem 2 we have

Theorem 4: If ii) in Theorem 3 is replaced by ii") and  $\Lambda$  is replaced by  $I_{2,0}^{-1}$ , then (S) has one and only one solution.

The next two criteria are based on Schauder's fixed point theorem.

Theorem 5: If i) - iv) hold and if, in addition,

vi)  $H_1, H_2$  can be extended to  $\hat{X}$  in such a way that  
 $\Lambda H_2, (-\Lambda L_2 \tilde{L}_1 + \tilde{L}_1) H_1$  are completely continuous operators of  $\hat{X}$  into  $X$ ;

vii) For some  $h_i > 0, (i = 1, 2)$  we have

$$(7) \quad \|H_i x\|_{X_i} \leq h_i, \quad x \in \hat{X}; \quad i = 1, 2$$

then (S) has solutions.

By vi) and (4) the operator  $T$  from  $\hat{X}$  into  $X$  is completely continuous. Moreover by (4) and (7)

$$|Tx|_{\hat{X}} \leq |\Lambda|h_2 + |-\Lambda L_2 \tilde{L}_1 + \tilde{L}_1|h_1, \quad x \in \hat{X}$$

so that  $T$  maps  $\hat{X}$  into a closed ball of  $X$ . By Schauder's Theorem (T) has solutions, hence (S) has solutions by Theorem 2.

Theorem 6: If the assumptions i) - iv) and vi) hold and if vii) is replaced by the weaker hypothesis

vii') There are three numbers  $\alpha > 0$ ,  $h_1(\alpha) > 0$ ,  $h_2(\alpha) > 0$ , such that

$$(7') \quad |H_1 x|_{X_1} \leq h_1(\alpha), \quad |x|_{\hat{X}} \leq \alpha, \quad i = 1, 2$$

and

$$(8) \quad |\Lambda|h_2(\alpha) + |-\Lambda L_2 \tilde{L}_1 + \tilde{L}_1|h_1(\alpha) \leq \alpha$$

then (S) has solutions.

If we define the two operators  $\hat{H}_1$ , ( $i = 1, 2$ )

$$\hat{H}_1 x = \begin{cases} H_1 x, & \text{for } |x|_X^\wedge \leq \alpha \\ H_1 \alpha |x|_X^{\wedge^{-1}} x, & \text{for } |x|_X^\wedge > \alpha \end{cases}$$

then  $\Lambda \hat{H}_2$  and  $(-\Lambda L_2 \tilde{L}_1 + \tilde{L}_1) \hat{H}_1$  are completely continuous.

Further from (7') we have

$$|\hat{H}_1 x|_{X_1} \leq h_1(\alpha), \quad x \in \hat{X}, \quad i = 1, 2$$

so that, by Theorem 5,

$$x = \Lambda \hat{H}_2 x + (-\Lambda L_2 \tilde{L}_1 + \tilde{L}_1) \hat{H}_1 x$$

has a solution  $\hat{x}$ . But

$$|\hat{x}|_X^\wedge \leq |\Lambda| h_2(\alpha) + |-\Lambda L_2 \tilde{L}_1 + \tilde{L}_1| h_1(\alpha)$$

hence by (8),  $|\hat{x}|_X^\wedge \leq \alpha$  and  $\hat{x}$  will be a solution of (T), i.e. of (S).

Remark 4. None of the preceding theorems requires that  $L_1$  be bounded.

Remark 5. When  $X_2$  and  $\mathcal{M}[L_1]$  are both  $n$ -dimensional then the existence of solutions of (S) under i), ii") (i.e. under i), ii'), see Remark 3) can be proved by using Brouwer's fixed point theorem in the euclidean  $n$ -dimensional space, if  $x = y + \tilde{L}_1 H_1 x$  has a unique solution for each  $y \in \mathcal{M}[L_1]$ . (cfr. H. Ehrmann, [1]).

5. To illustrate the preceding sections by an example let us consider a general boundary value problem for an ordinary differential equation

$$(9) \quad dx/dt = f(t, x)$$

where  $t$  is a real variable in a given compact interval  $\Delta = [a, b]$  of the real line  $E^1$ , and  $f(t, u)$  is an  $n$ -vector function of  $t \in \Delta$ ,  $u \in E^n$ , the euclidean  $n$ -dimensional space. Beside  $E^n$  we shall also consider the linear spaces  $C^1(\Delta)$ ,  $AC(\Delta)$ ,  $C(\Delta)$  and  $L^1(\Delta)$  of  $n$ -vector functions whose components are of class  $C^1$ , absolutely continuous, continuous or  $L$ -integrable on  $\Delta$ , respectively.

The most general bounded linear operator of  $C(\Delta)$  into  $E^n$  is represented by

$$x \rightarrow \int_{\Delta} dFx, \quad x \in C(\Delta)$$

where  $F = F(t)$  is an  $n \times n$  matrix function of  $t$  of bounded variation on  $\Delta$ , and the integral is a Riemann-Stieltjes' one. Denoting by  $h(u)$  a continuous  $n$ -vector function of  $u \in E^n$  we shall associate with (9) the "boundary" condition

$$(10) \quad \int_{\Delta} dFx = h(x)$$

This is an actual boundary condition for particular  $F$ . For instance if

$$F = \begin{cases} I, & \text{for } t = a \\ 0, & \text{for } a < t < b \\ -I, & \text{for } t = b \end{cases}$$

where  $I, 0$  are the identity and the zero  $n \times n$  matrix, respectively, (10) reduces to

$$x(a) - x(b) = h(x).$$

To interpret (9) (10) as a problem of type (S) we shall replace  $f(t, u)$  by  $g(t, u) = f(t, u) - A(t)u$ , where  $A(t)$  is an  $n \times n$  matrix function of  $t \in \Delta$ , so that (9) (10) may be written as

$$(E) \quad \begin{cases} dx/dt - A(t)x = g(t, x) \\ \int_{\Delta} dFx = h(x) \end{cases}$$

If  $f$  is continuous we shall assume  $A$  (hence  $g$ ) also continuous and we shall look for solutions of (E) in the classical sense,  $x \in C^1(\Delta)$ . In this case

$$X = C^1(\Delta), \quad X_1 = C(\Delta), \quad X_2 = \mathbb{R}^n.$$

If  $f$  is subjected to Carathéodory type assumptions, taking a matrix  $A$  which is  $L$ -integrable on  $\Delta$ , we shall look for solutions of (E) in the

Caratheodory sense,  $x \in AC(\Delta)$ . In this case

$$X = AC(\Delta), \quad X_1 = L^1(\Delta), \quad X_2 = E^n.$$

In both cases  $L_1, L_2, H_1, H_2$  are defined by

$$L_1 x = [d/dt - A(t)]x, \quad L_2 x = \int_{\Delta} dFx$$

$$H_1 x = g(t, x), \quad H_2 x = h(x)$$

and  $H_1, H_2$  can be obviously extended from  $X$  to  $\hat{X} = C(\Delta)$ .

The null space  $\mathcal{N}[L_1]$  of  $L_1$  will be the space of the solutions  $y$  of the homogeneous equation

$$dy/dt - A(t)y = 0 \quad .$$

If we denote by  $Y(t)$  any fundamental matrix of this equation we may define

$$\tilde{L}_1 x = \int_a^t Y(\epsilon) Y^{-1}(s) x(s) ds, \quad x \in X_1$$

and we see that  $L_1 \tilde{L}_1 x = x$ ,  $x \in X_1$ , so that the assumption 1) of Section 1 is verified. (Of course  $\tilde{L}_1$  is no (left) inverse of  $L_1$ , since  $\tilde{L}_1 L_1 x = x - \bar{y}$ ,  $x \in X$ , where  $d\bar{y}/dt - A(t)\bar{y} = 0$ ,  $\bar{y}(a) = x(a)$ ).

$\mathcal{M}[L_1]$  and  $X_2 = E^n$  are isomorphic and, denoting by  $C$  any  $n \times n$  matrix (singular or not), all the linear operators  $\Lambda$  of  $X_2 = E^n$  into  $\mathcal{M}[L_1]$  can be represented by

$$\Lambda = Y(t)C \quad .$$

Consequently, if we put

$$\int_{\Delta} dF Y(\tau) = D$$

the system  $(S_A)$  of Section 1 becomes here

$$\left\{ \begin{array}{l} x = Y(t)Ch(x) + (-Y(t)C \int_{\Delta} dF \int_a^t Y(\tau)Y^{-1}(s)g(s, x(s))ds + \\ \quad + \int_a^t Y(t)Y^{-1}(s)g(s, x(s))ds) \\ (DC - I) (h(x) - \int_{\Delta} dF \int_a^t Y(\tau)Y^{-1}(s)g(s, x(s))ds) = 0 \end{array} \right.$$

or, more simply

$$(E_C) \quad \left\{ \begin{array}{l} x = Y(t)Ch(x) + \int_{\Delta} G(t, s)g(s, x(s))ds \\ (DC - I) (h(x) - \int_{\Delta} dF \int_a^t Y(\tau)Y^{-1}(s)g(s, x(s))ds) = 0 \end{array} \right.$$

where  $G(t, s)$  is a Green matrix defined by

$$G(t,s) = \begin{cases} -Y(t)C \int_s^b dFY(\tau)Y^{-1}(s) + Y(t)Y^{-1}(s), & a \leq s < t \leq b \\ -Y(t)C \int_s^b dFY(\tau)Y^{-1}(s), & a \leq t \leq s \leq b \end{cases}$$

The assumption ii) of Section 2 means that for some  $n \times n$  constant matrix  $C$

$$(11) \quad (DC - I) (h(x) - \int_a^b dF \int_a^t Y(\tau)Y^{-1}(s)g(s,x(s))ds) = 0, \quad x \in X$$

If we denote by  $D^\#$  the pseudoinverse matrix of  $D$ , which is uniquely defined by

$$D^\# D D^\# = D^\#, \quad (D^\# D)^* = D^\# D, \quad D D^\# D = D, \quad (D D^\#)^* = D D^\#$$

(where  $*$  denotes the transpose) (see R. Penrose, Proc. Cambridge Philos. Soc., 51, part 3 (1955), 406-413; C. A. Desoer, B. H. Whalen, J. SIAM, 11 (1963), 442-447) we see that

$$\mathcal{N}_{[DD^\# - I]} \supset \mathcal{N}_{[DC - I]}$$

(In fact if  $x \in \mathcal{N}_{[DC - I]}$ , i.e.  $x = DCx$ , we have  $DD^\#x = DD^\#DCx = DCx = x$ , i.e.  $x \in \mathcal{N}_{[DD^\# - I]}$ ).

Therefore

$$(DD^{\#} - I) (h(x) - \int_{\Delta} dF \int_a^T Y(\tau) Y^{-1}(s) g(s, x(s)) ds) = 0, \quad x \in X$$

is the weakest assumption of type (11).

In particular, the assumption ii') (or the equivalent ii'') means that  $D$  has a true inverse  $D^{-1}$ , and in this case

$$L_{2,0}^{-1} = Y(t) D^{-1}.$$

This case could be called "non-resonant" extending the terminology used for the special boundary value problem of finding the periodic harmonic solutions of  $dx/dt - A(t)x = g(t, x)$ .

6. If  $g(t, u)$  is continuous for  $(t, u) \in \Delta \times E^n$ , both two operators  $H_1 x = h(x)$ ,  $H_2 x = g(t, x)$  will map  $\hat{X} = C(\Delta)$  into  $X = C^1(\Delta)$ .

Introducing the uniform norm into  $\hat{X} = C(\Delta)$  this is a Banach space so that iii) is satisfied.

Since  $A = Y(t)C$  it is readily seen that iv) is also satisfied. To comply with v) we may assume

$$\|h(u') - h(u'')\| \leq \lambda_2 \|u' - u''\| \quad u', u'' \in E^n$$

$$\|g(t, u') - g(t, u'')\| \leq \lambda_1 \|u' - u''\|, \quad u', u'' \in E^n$$

where  $\| \cdot \|$  is the euclidean norm, and

$$(12) \quad \lambda_2 \sup_{t \in \Delta} |Y(t)C| + (b - a) \lambda_1 \sup_{t, s \in \Delta} |G(t, s)| < 1$$

and this, together with (11), will insure the existence of at least one solution of (E) in class  $C^1(\Delta)$ , by virtue of Theorem 3. In the non-resonant case, (12) with  $C$  replaced by  $D^{-1}$ , will also insure the uniqueness of such solution, by Theorem 4.

If  $g(t, u)$  is of Caratheodory type then we may use Theorem 3 and Theorem 4 to obtain existence and uniqueness criteria for the solutions of (E) in class  $AC(\Delta)$ . (See G. Santagati, [1]).

For some applications of Theorem 5 and 6 to the problem (E) we refer to the work by the present author (R. Conti, [1]) and G. Pulvirenti [1].

In the non-resonant case the existence of a unique solution of (E) in class  $AC(\Delta)$  can be proved under conditions which are more general than Lipschitz's ones (See P. Santoro, [1]). To complete our references to problem (E), we shall mention the work by W. M. Whyburn, [1], [2] and by G. Santagati [1], who studied the continuous dependence of the solution of (E) on the data  $A, F, h, g$ .

References

- R. Conti, [1] Annali di Mat., (4) 57 (1962) 49-62; [2], Accad. Naz. dei Lincei, Rend. Cl. Sci. fis. mat. nat., (8) 32 (1962), 495-498.
- H. Ehrmann, [1] Arch. Rat. Mech. Anal., 7 (1961), 349-358;  
[2] On implicit function theorems and the existence of solutions of nonlinear equations, The University of Wisconsin, Math. Res. Center, U.S. Army, Technical Summary Report No. 343, Aug. 1962.
- I. I. Kolodner, [1] Equations of Hammerstein type in Hilbert spaces, The University of New Mexico, Department of Math., Technical Report, No. 47, Nov., 1963.
- M. A. Krasnosel'skii, [1] Uspehi Mat. Nauk, IX 3(61) (1954), 57-114;  
AMS Transl., Ser. 2, 10 (1958), 344-409; [2] Topological methods in the theory of non-linear integral equations; Moscow, GITTL, 1956; Pergamon Press, 1964.
- G. Pulvirenti, [1] Le Matematiche (Catania), 16 (1961) 27-42.
- G. Santagati, [1] Annal. d Mat., (4) 62 (1963), 335-370.
- P. Santoro, [1] Accad. Naz. dei Lincei, Rend. Cl. Sci. fis. Mat. Nat., (8) 32 (1962) 903-906.
- W. M. Whyburn, [1] Trans. AMS, 32 (1929), 502-508; [2] Proc. of the Conference on D. E., University of Maryland, College Park (1955) 1-20.